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# Linear Forms in Logarithms on Elliptic Curves

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**概要**

Diophantine 近似という言葉で表される内容には、主に超越数を代数的数で近似するものと、無理数を有理数で近似する方法とがある。ここでは、楕円曲線の代数点のペエ関数についての逆像の点の、代数的係数の 1 次結合の絶対値に対する下からの評価という近似について、係数の高さについての最良評価をフランスの S. David と共同で得たことについて、報告する。

これは少なくとも１９７７年の M. Anderson の論文 [An] に載っている一つの問題に対する答えとなるが、実際にはもっと前から、通常の対数一次形式と同じ評価を得るという問題として考えられていたようである。

平田は１９９１年にこの最良評価の少し手前のものまで到達していた [Hi2]（これはアーベル多様体など、可換代数群上で OK ）。これはちょうどヘルシンキの ICM コングレスで提出されていたG. Chudnovsky の予想を解決するものになったが、今回の話は楕円曲線の場合にその改良に至ったということ、即ち楕円曲線の有理点の 1 次結合については、係数の高さに関しての初めての最良評価を得たという報告である。具体的には $(\log B + \log(DE) + \log\log V + h)$ の項を、指数 1 という最良のものまで、落とせたということになる。

これは、定数を計算しておけば、たとえば楕円曲線の整数点の計算などにも応用される。

改良の鍵は、別の近似に対する G. Chudnovsky [Ch] の考え方から思い付いたもので、ペエ関数による楕円曲線上の有理点の記述を楕円曲線上のフォーマルグループによるものに直し、楕円対数関数を直接証明に使うという方法である。いわば、E 関数のかわりに G 関数を考えるという、変換を施したわけである。

## Introduction

Let $K$ be an algebraic number field of degree $D$ over the rational number field $\mathbb{Q}$. We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $k$ be a rational integer $\geq 1$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be $k$ elliptic curves defined over $K$. We assume that these curves are defined by Weierstraß' equations, normalized as follows :

$$y^2 = 4x^3 - g_{2,i}x - g_{3,i} \; : \quad g_{2,i}, g_{3,i} \in K, \quad 1 \leq i \leq k.$$

We denote by $\wp_i$, for $1 \leq i \leq k$ (resp. $\sigma_i$, for $1 \leq i \leq k$), the Weierstraß' elliptic functions (resp. the Weierstraß' sigma functions), associated with the underlying period lattice $\Lambda_i = \omega_{1,i}\mathbb{Z} + \omega_{2,i}\mathbb{Z}$, $1 \leq i \leq k$.

For each $1 \leq i \leq k$, let $u_i \in \mathbb{C}$ satisfy

$$\gamma_i := (\sigma_i^3(u_i), \sigma_i^3(u_i)\wp_i(u_i), \sigma_i^3(u_i)\wp_i'(u_i)) \in \mathcal{E}_i(\overline{\mathbb{Q}}).$$

When $u_i$ is a pole of $\wp_i$, we consider $\gamma_i = (0, 0, 1)$.

Such complex numbers $u_1, \ldots, u_k$ are called elliptic logarithms (of rational points).

Thus, clearly, any point in the period lattice is an elliptic logarithm.

Let $N \geq 1$ be an integer and $P = (x_0, \ldots, x_N) \in \mathbb{P}^N(\overline{\mathbb{Q}})$. We introduce the absolute logarithmic projective height on $\mathbb{P}^N$. Let $L$ be a number field containing all coordinates of the point $P$. Put

$$h(P) = \frac{1}{[L : \mathbb{Q}]} \sum_v n_v \log(\max\{|x_0|_v, \ldots, |x_N|_v\}),$$

where $v$ runs over the set of absolute values of $L$ which are normalised such that for all $x \in L, x \neq 0$, we have $\sum_v n_v \log |x|_v = 0$ and $\sum_{v|\infty} n_v = d$. Here, we denote by $n_v = [K_v : \mathbb{Q}_v]$ the local degree at each $v$. Because of the extension formula, it is well known that $h(P)$ is independent of the choice of the field $L$, and the product formula ensures on the other hand that the definition does not depend on the choice of projective coordinates of $P$.

The study of linear forms in elliptic logarithms derives from an analogy with the theory of linear forms in usual logarithms, simply by viewing the Weierstraß' elliptic $\wp$-function with algebraic invariants as an exponential map of an elliptic curve (*i.e.* a commutative algebraic group) defined over a number field.

A basic question is to ask whether non-zero elliptic logarithms of rational points are transcendental. An answer was first given by C. L. Siegel in 1932 (*see* [Sie]). For $k = 1$, we write $u = u_1$, $\Lambda = \Lambda_1$, and $\wp = \wp_1$, in our notations set above. He showed that there exists at least one element of $\Lambda$ which is transcendental over $\mathbb{Q}$. If $\wp$ has complex multiplication, it is well known that

the ratio of two non-zero elements of $\Lambda$ belongs to the corresponding quadratic imaginary field $\mathfrak{K}$. Thus, in the case of complex multiplication, Siegel's result implies that any non-zero element in $\Lambda$ is transcendental. In 1937, Th. Schneider proved more generally (*confer* [Sc1]) that any elliptic logarithm $u$ is either zero or transcendental without any hypothesis of complex multiplication. Now consider the case $k = 2$ with $\mathcal{E}_1 = \mathcal{E}_2$, $\wp := \wp_1 = \wp_2$. Th. Schneider also showed that the quotient of two elliptic logarithms $u_1, u_2$ is either transcendental or rational if $\wp$ has no complex multiplication, and either transcendental or an element of $\mathfrak{K}$ if $\wp$ has complex multiplication over $\mathfrak{K}$. Indeed, in both CM and non-CM cases, for $u_1 \neq 0, u_2 \neq 0$, his result yields that a necessary and sufficient condition for the transcendence of $\frac{u_1}{u_2}$ is the algebraic independence of the two functions $\wp(u_1 z)$ and $\wp(u_2 z)$ (*see* [Sc3]).

A. Baker proved in 1970 (*confer* [Ba1]), using the method he developped for the study of linear forms in usual logarithms (*see* [Ba3]), when $k = 2$, $u_1 \in \Lambda_1$ and $u_2 \in \Lambda_2$, that the linear form $\beta_1 u_1 + \beta_2 u_2$ with algebraic coefficients $\beta_1, \beta_2$ is either zero or transcendental (see also related results together with quasi-periods and $2\pi i$ by S. Lang, J. Coates and by D. W. Masser, mentioned in [Ma5]).

In 1975, D. Masser succeeded in a generalization to arbitrary $k$ elliptic logarithms $u_1, \ldots, u_k$ when $\mathcal{E}_1 = \cdots = \mathcal{E}_k$, provided that $\wp := \wp_1 = \cdots = \wp_k$ has complex multiplication over $\mathfrak{K}$ : if $u_1, \ldots, u_k$ are linearly independent over $\mathfrak{K}$, then $1, u_1, \ldots, u_k$ are linearly independent over $\overline{\mathbb{Q}}$ (Chapter 7 with Appendix 3 of [Ma1]). This was extended in 1980, to the non-CM case by D. Bertrand and D. Masser : suppose that $\wp$ has no complex multiplication and that $u_1, \ldots, u_k$ are linearly independent over $\mathbb{Q}$. Then $1, u_1, \ldots, u_k$ are linearly independent over $\overline{\mathbb{Q}}$ (*confer* [Be-Ma1]).

Generalizations in the abelian case were treated by Th. Schneider (*see* [Sc2]) in 1941 for abelian integrals, more generally by S. Lang and by D. Masser (*confer* [La1], [Ma2], [La2], [Ma3], [Ma4]). D. Masser proved the linear independence of "abelian" logarithms over $\overline{\mathbb{Q}}$ under a hypothesis of complex multiplication (with a quantitative version of exponential magnitude : see below). The non-CM case was presented in 1980 by D. Bertrand and D. Masser (*see* [Be-Ma2]); they however needed real multiplication.

Let us consider the linear independence problem of elliptic logarithms without the simplifying hypothesis $\mathcal{E}_1 = \cdots = \mathcal{E}_k$, nor assuming complex multiplication. More generally, consider the corresponding problem on a connected commutative algebraic group defined over a number field. The linear independence over $\overline{\mathbb{Q}}$ of 1 and "generalized abelian" logarithms was proven by G. Wüstholz in 1989 (*confer* [Wü]), where we can deduce all qualitative results mentioned above as corollaries.

From now on, we give an account of the history of quantitative estimates.

In 1951, N. I. Fel'dman showed a Diophantine approximation measure of

an elliptic logarithm by an algebraic number. Precisely, it concerns the case $k = 1$, $u := u_1 \neq 0$ in our notations above. Write $h(\beta) := h(1, \beta)$ if $\beta$ is algebraic. Let $B$ be a real number $\geq 3$. He proved that there exists an effective constant $c > 0$ which is independent of $B$ such that for any $\beta \in \overline{\mathbb{Q}}$ with $h(\beta) \leq \log B$ we have

$$\log|u - \beta| \geq -\log B \cdot \exp\{c(\log\log B)^{1/2}\};$$

he refined the estimate for a non zero period $u \in \Lambda := \Lambda_1$ to obtain

$$\log|u - \beta| \geq -c \cdot \log B \cdot (\log\log B)^4.$$

The case of a quotient of two non-zero elliptic logarithms was also treated by him (*confer* [Fe1], [Fe2], [Fe3]) (in fact, he used a classical height, but it can be translated to the logarithmic height; see the relation between various heights in [Wa]).

Let $\mathcal{L}(\mathbf{z}) = \beta_0 z_0 + \cdots + \beta_k z_k$ be a non zero linear form on $\mathbb{C}^{k+1}$ with coefficients in $K$. We write $\mathbf{v} = (1, u_1, \ldots, u_k)$. Let $B$ be a real number satisfying $B \geq e$.

A. Baker proved a positive lower bound of $|\mathcal{L}(\mathbf{v})|$ in 1970 (*see* [Ba2]) for $k = 2, \mathcal{E}_1 = \mathcal{E}_2, u_1, u_2 \in \Lambda := \Lambda_1 = \Lambda_2$ and $\beta_0 \neq 0$. D. Masser showed in [Ma1] the following estimate in 1975 for arbitrary $k$, $\mathcal{E}_1 = \cdots = \mathcal{E}_k$ and $\beta_0 = 0$ under a hypothesis of complex multiplication over $\mathfrak{K}$ ; assume that $u_1, \ldots, u_k$ are linearly independent over $\mathfrak{K}$. For any $\epsilon > 0$, there exists an effective constant $c > 0$ which depends on $\epsilon$ and other data but independent of $B$ such that for any $\beta_1, \ldots, \beta_k \in K$ satisfying $h(\beta_i) \leq \log B$ ; $1 \leq i \leq k$, we have $\log|\mathcal{L}(\mathbf{v})| \geq -c \cdot B^{\epsilon}$ (see also abelian cases in [La2], [Ma2], [Ma3], [Ma4] ; the estimates in [Ma2] and [Ma4] are of the same magnitude). Also assuming complex multiplication, J. Coates and S. Lang [Co-La] refined this estimate in 1976, actually in the abelian case, to get $\log|\mathcal{L}(\mathbf{v})| \geq -c \cdot (\log B)^{8k+6+\epsilon}$. In 1977, M. Anderson refined this estimate and proved in the not necessarily homogeneous case but still assuming complex multiplication on elliptic curves : $\log|\mathcal{L}(\mathbf{v})| \geq -c \cdot \log B \cdot (\log\log B)^{k+1+\epsilon}$, where $h(\beta_0) \leq \log B$, and $\log B \geq e$. Some related results were treated by W. D. Brownawell and D. Masser in [Bro-Ma], by E. Reyssat (*see* [Re]) and by Kunrui Yu (*confer* [Yu]).

In 1988 P. Philippon and M. Waldschmidt showed the first such estimate without any hypothesis of complex mulplication (*see* [Ph-Wa]). Let us denote $\mathcal{W} = \ker(\mathcal{L})$. Suppose that for any connected algebraic subgroup $\mathbb{G}'$ of $\mathbb{G} := \mathbb{G}_a \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ with $T_{\mathbb{G}'}(\mathbb{C}) \subset \mathcal{W}$ we have $\mathbf{v} \notin T_{\mathbb{G}'}(\mathbb{C})$ (here we write $T_{\mathbb{G}'}(\mathbb{C})$ the tangent space of $\mathbb{G}$ at the origin and $\mathbb{G}_a$ stands for the additive group). Let $B$ be a real number satisfying $\log B \geq \max\{1,\ h(\beta_i)\ ;\ 0 \leq i \leq k\}$. Then they obtained a lower bound of the form

$$|\mathcal{L}(\mathbf{v})| \geq \exp\left(-c \cdot (\log B)^{k+1}\right).$$

They did not assume $\mathcal{L}(\mathbf{v}) \neq 0$ as was often done; thus we can deduce also qualitative linear independence or transcendence results from this quantitative one (such a lower bound clearly implies that $\mathcal{L}(\mathbf{v}) \neq 0$). In fact, they proved a result in the general case where $\mathbb{G}$ is any connected commutative algebraic group. This estimate was refined by the second author in 1991 (*see* [Hi1], [Hi2]) with $\log B \geq e$ to get

$$\log|\mathcal{L}(\mathbf{v})| \geq -c \cdot \log B \cdot (\log\log B)^{k+1}$$

also in the case of connected commutative algebraic group, relying upon an idea originally due to N. Feld'man (*confer* [Fel]) also used in E. Reyssat's work (*see* [Re]) but by introducing a "redundant variable".

The first author then gave in 1995 (*confer* [Da]) a completely explicit version in the elliptic case of this result, with $c$ made explicit as a function of all given data. Here, the dependence of $|u_i|$ with $1 \leq i \leq k$ is better than the previous results when these quantities are small. In 1998, M. Ably (*see* [Ab]) showed in the elliptic case an estimate of the form

$$\log|\mathcal{L}(\mathbf{v})| \geq -c \cdot \log B$$

under a hypothesis of complex multiplication. For this purpose, he generalized Fel'dman's polynomials to quadratic fields and studied their properties. He was thus the first to obtain the optimal estimate in the elliptic case, albeit with the extra hypothesis of complex multiplication. A little later, in 1999 a special case related with periods and quasi-periods of an elliptic function was treated by S. Bruiltet (*see* [Bru]), where one part corresponds in fact to a statement announced by G. V. Chudnovsky in 1984 (*confer* [Ch]). We would also like to mention a work by E. Gaudron, which aims to provide an estimate of the same optimal shape, *i.e.* $-c \cdot \log B$ for any commutative algebraic group, by studying the arithmetic properties of infinitesimal neighborhoods of the origin on suitable integral models.

Our contribution basically originates from an idea of G. Chudnovsky, which says that local parameters have better arithmetic properties than the complex uniformization, though they do not have a good analytic behaviour. We therefore build on his idea of "variable change" (see Chapter 8 on algebraic independence measure of [Ch]) to the case of elliptic logarithms, which are not necessarily in the period lattice, and we work with the parameters coming from the so-called formal group (*see eg.* chapter IV of [Sil]).

### New result

We put $\tau_i = \frac{\omega_{2,i}}{\omega_{1,i}}, 1 \leq i \leq k$. There is no restriction to assume that $\tau_i$ belongs to the upper half plane $\mathfrak{H}$, even, to the usual fundamental domain $\mathfrak{F}$ of $\mathfrak{H}$ by

the action of $SL_2(\mathbb{Z})$ ; for this, we choose a suitable basis of $\Lambda_i$ and this does not change the invariants $g_{2,i}, g_{3,i}, 1 \leq i \leq k$.

We denote by $h = \max\{1, h(1, g_{2,i}, g_{3,i}) \ ; \ 1 \leq i \leq k\}$ the height of our elliptic curves.

We also denote by $\hat{h}(\gamma_i)$ the Néron–Tate height of $\gamma_i$ defined as in [Sil], namely, $\hat{h}(\gamma_i) = \lim_{n\to\infty} \frac{h(n\gamma_i)}{n^2}$.

Finally we put $\mathbb{G} = \mathbb{G}_a \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ which is a connected commutative algebraic group. Write $T_{\mathbb{G}}(\mathbb{C})$ for the tangent space of $\mathbb{G}$ at the origin which we shall identify with $\mathbb{C}^{k+1}$. We denote by $T_{\mathbb{G}'}(\mathbb{C})$ the tangent space at the origin of an algebraic subgroup $\mathbb{G}'$ of $\mathbb{G}$.

Now we present our result.

**Theorem (with S. DAVID)** [Da-Hi] *There exists an effective funcion $C > 0$ of $k$, with the following property. Let $\mathcal{L}(\mathbf{z}) = \beta_0 z_0 + \cdots + \beta_k z_k$ be a non zero linear form on $\mathbb{C}^{k+1}$ with coefficients in $K$, we put $\mathcal{W} = \ker(\mathcal{L})$; let moreover $u_1, \ldots, u_k$ be complex numbers such that $\gamma_i = (1, \wp_i(u_i), \wp_i'(u_i)) \in \mathcal{E}_i(K) \subset \mathbb{P}^2(K)$ if $u_i \notin \Lambda_i$, and $\gamma_i = (0,0,1)$ if $u_i \in \Lambda_i$ for $1 \leq i \leq k$. We write $\mathbf{v} = (1, u_1, \ldots, u_k)$. Let $B$, $E$, $V_1, \ldots, V_k$ be real numbers satisfying the following conditions :*

$$\log B \geq \max\{1, h(\beta_i) \ ; \ 0 \leq i \leq k\}$$

$$V_1 \geq \cdots \geq V_k$$

$$\log V_i \geq \max\left\{e, \hat{h}(\gamma_i) \ , \ \frac{|u_i|^2}{D|\omega_{1,i}|^2 \Im m\, \tau_i}\right\}, \ 1 \leq i \leq k$$

$$e \leq E \leq \min\left\{\frac{|\omega_{1,i}|\,(\Im m\, \tau_i \cdot D \log V_i)^{\frac{1}{2}}}{|u_i|} \ ; \ 1 \leq i \leq k\right\}.$$

*Suppose that for any connected algebraic subgroup $\mathbb{G}'$ of $\mathbb{G}$ with $T_{\mathbb{G}'}(\mathbb{C}) \subset \mathcal{W}$, we have $\mathbf{v} \notin T_{\mathbb{G}'}(\mathbb{C})$.*

*Then we have $\mathcal{L}(\mathbf{v}) \neq 0$ and*

$$\log|\mathcal{L}(\mathbf{v})| \geq \ -C \cdot D^{2k+2} \times (\log E)^{-2k-1} (\log B + \log(DE) + h + \log\log V_1)$$

$$\times (\log(DE) + h + \log\log V_1)^{k+1} \prod_{i=1}^{k} (h + \log V_i)\,.$$

Thus we obtain here a lower bound of the form

$$\log|\mathcal{L}(\mathbf{v})| \geq \ -c \cdot \log B$$

without any hypothesis of complex multiplication.

## REFERENCES


[Ab] M. Ably, *Formes linéaires de logarithmes de points algébriques sur une courbe elliptique de type CM*, Ann. de l'Institut Fourier (to appear).

[An] M. Anderson, *Inhomogeneous linear forms in algebraic points of an elliptic function*, Transcendence Theory : Advances and Applications, Academic Press (1977), 121-144.

[Ba1] A. Baker, *On the periods of the Weierstraß $\wp$-function*, Symposia Math, IV, INDAM Rome 1968, Academic Press (1970), 155-174.

[Ba2] A. Baker, *An estimate for the $\wp$-function at an algebraic point*, Amer. J. Math. 92 (1970), 619-622.

[Ba3] A. Baker, *Transcendental Number Theory (Cambridge Math. Library series)*, Cambridge Univ. Press (1975).

[Be-Ma1] D. Bertrand and D. W. Masser, *Linear Forms in Elliptic Integrals*, Inventiones Math. 58 (1980), 283-288.

[Be-Ma2] D. Bertrand and D. W. Masser, *Formes linéaires d'intégrales abéliennes*, C. R. Acad. Sc. Paris, t. 290, Série A (1980), 725-727.

[Bro-Ma] W. D. Brownawell and D. W. Masser, *Multiplicity estimates for analytic functions I*, J. reine angew. Math. 314 (1980), 200-216.

[Bru] S. Bruiltet, *D'une mesure d'approximation simultanée à une mesure d'irrationalité : $\Gamma(1/4)$ et $\Gamma(1/3)$*, preprint.

[Ch] G. V. Chudnovsky, *Contributions to the theory of transcendental numbers*, Amer. Math. Soc. Math. Surveys Monographs 19 (1984).

[Co-La] J. Coates and S. Lang, *Diophantine approximation on Abelian varieties with complex multiplication*, Inventiones Math. 34 (1976), 129-133.

[Da] S. David, *Minorations de formes linéaires de logarithmes elliptiques*, Mémoires, Nouvelle série 62, Supplément au Bulletin de la Soc. Math. de France, Tome 123, Fascicule 3 (1995).

[Da-Hi] S. David et N. Hirata-Kohno, *Formes linéaires de logarithmes elliptiques*, preprint.

[Fe1] N. I. Fel'dman, *Approximation of certain transcendental numbers II : The approximation of certain numbers associated with the Weierstraß $\wp$-function*, Izv. Akad. Nauk. SSSR. Ser. Mat. 15 (1951), 153-176 (Amer. Math. Trans. Ser. 2, 59, 1966, 246-270).

[Fe2] N. I. Fel'dman, *Simultaneous approximation of the periods of an elliptic function by algebraic numbers*, Izv. Akad. Nauk. SSSR, Ser. Mat. 22 (1958), 563-576 (Amer. Math. Trans. Ser. 2, 59, 1966, 271-284).

[Fe3] N. I. Fel'dman, *An elliptic analogue of an inequality of A. O. Gel'fond*, Trudy. Moskov, 18 (1968), 65-76 (Trans. Moscow Math. Soc. 18, 1968, 71-84).

[Hi1] N. Hirata-Kohno, *Formes linéaires d'intégrales elliptiques*, Sém. de Théorie des Nombres, Paris, 1988/89, Progress in Math. 91, C. Goldstein éd. Birkhäuser (1990), 1-23.

[Hi2] N. Hirata-Kohno, *Formes linéaires de logarithmes de points algébriques sur les groupes algébriques*, Inventiones Math. 104 (1991), 401-433.

[La1] S. Lang, *Diophantine approximation on toruses*, Amer. J. Math.86 (1964), 521-533.

[La2] S. Lang, *Diophantine approximation on Abelian varieties with complex multiplication*, Advances in Math. 17 (1975), 281-336.

[Ma1] D. W. Masser, *Elliptic Functions and Transcendence*, Lecture Notes in Math. 437, Springer (1975).

[Ma2] D. W. Masser, *Linear forms in algebraic points of Abelian functions I*, Math. Proc. Cambridge Phil. Soc. 77 (1975), 499-513.

[Ma3] D. W. Masser, *Linear forms in algebraic points of Abelian functions II*, 79 (1976), 55-70.

[Ma4] D. W. Masser, *Linear forms in algebraic points of Abelian functions III*, Proc. London Math. Soc. 33 (1976), 549-564.

[Ma5] D. W. Masser, *Some vector spaces associated with two elliptic functions*, Transcendence Theory : Advances and Applications, Academic Press (1977), 101-120.

[Ph-Wa] P. Philippon et M. Waldschmidt, *Formes linéaires de logarithmes sur les groupes algébriques commutatifs*, Illinois J. Math. 32 (1988), 281-314.

[Re] E. Reyssat, *Approximation algébrique de nombres liés aux fonctions elliptiques et exponentielle*, Bull. Soc. Math. France 108 (1980), 47-79.

[Sc1] Th. Schneider, *Arithmetische Untersuchungen elliptischer Integrale*, Math. Annalen 113 (1937), 1-13.

[Sc2] Th. Schneider, *Zur Theorie der Abelschen Funktionen und Integrale*, J. reine angew. Math. 183 (1941), 110-128.

[Sc3] Th. Schneider, *Einführung in die transzendenten Zahlen*, Springer (1957).

[Sie] C. L. Siegel, *Über die Perioden elliptischer Funktionen*, J. reine angew. Math. 167 (1932), 62-69.

[Sil] J. H. Silverman, *The arithmetic of elliptic curves*, GTM 106 (Springer) (1986).

[Wa] M. Waldschmidt, *Nombres transcendants et groupes algébriques*, Astérisque 69/70 (1979).

[Wü] G. Wüstholz, *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen*, Ann. Math.129 (1989), 501-517.

[Yu] Kunrui Yu, *Linear forms in elliptic logarithms*, J. Number Theory 20 (1985), 1-69.